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On the surface stability of liquid conductors in electromagnetic shaping

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In a process involving electromagnetic shaping, a high-frequency electromagnetic field is used to deform a liquid conductor into a required shape. This is particularly relevant to processes such as levitation melting. In this paper the stability of such configurations are investigated. The second variation of an appropriate energy functional is derived whose minimum states correspond to stable configurations, thus providing a stability criterion. As an example, this is applied to the shaping of a levitated cylinder of circular cross-section and to an almost spherical axisymmetric shape. In both cases we find that these shapes are unstable. We then consider enclosing the entire shaping device in a metal shield, thus preventing the escape of the magnetic field. It is then shown that in general the shield has a stabilizing effect, whose exact nature depends on the topology of the liquid shape and on the field structure on its surface. This differing behaviour is discussed for two-dimensional spherical and toroidal shapes.

1. Introduction

In the various processes involving liquid metals there arises the problem of controlling the equilibrium of the mass of metal or of its surface by a high-frequency oscillating magnetic field. The interest in electromagnetic shaping problems arises mainly in two areas: continuous casting and levitation melting. Continuous casting is a standard process in the metallurgical industry, and electromagnetic forces may be advantageously used to shape the vertically falling column before its outer layer solidifies. Experimental work in this field has been carried out in Grenoble (Etay 1980; Garnier & Etay 1982). In levitation melting, on the other hand, a piece of solid metal is placed in a basket of coils, which produce a high-frequency field sufficiently strong to levitate the sample. The eddy currents induced in the metal are usually strong enough to melt it. The resulting shape of the molten metal will then be determined by the shape of the applied field. This melting process was first suggested by Muck (1923) in a German patent, but it was some years later that the first experimental work was published by Okress *et al.* (1952). Many levitation devices were developed by Polonis (Polonis & Parr 1954). The main advantage of levitation melting over the conventional crucible melting is that the liquid metal does not come into contact with the crucible wall, thereby avoiding any undesired contamination.

From a theoretical point of view, these two shaping problems are governed by the same equations, (1.2) and (1.3) below, and several studies have been carried out to determine the resulting liquid shape for a given external current distribution j_0. This is usually referred to as the *direct problem*.

In general, the direct problem can be solved either by considering equilibrium equations at the interface or by minimizing the appropriate energy functional introduced by Sneyd & Moffatt (1982), the main advantage of this method being that the resulting shapes are then mechanically stable. Several problems have been studied using either of these methods. We mention the work done by Brancher & Sero Guillaume (1983), Brancher, Etay & Sero Guillaume (1983) for the shaping of jets, and that of Mestel (1982), Sneyd & Moffatt (1982), Etay, Gagnoud & Garnier (1986) for the electromagnetic levitation.

The *inverse problem*, on the other hand, consists of determining the exterior field, and therefore the external currents, for which the liquid metals takes on a given desired shape. In spite of the obvious importance of the inverse problem in the design of specific coil arrangements, not much work has been dedicated to it. It was not until 1989 that Henrot, Pierre & Brancher provided a constructive method to solve the inverse problem for two-dimensional configurations (with direct application to the shaping of jets), and this was extended to the three-dimensional case by the present author (Felici 1991).

There now arises the question of whether these given shapes are stable. In this paper we investigate the mechanical stability of the liquid using the energy functional mentioned above. Sneyd & Moffatt (1982) derived its first variation and showed that its stationary 'points' correspond to equilibrium configurations. Here we shall pay some attention to the first variation of the magnetic term of this energy functional, not only to introduce the notation that we will need for the second variation, but also to emphasize a minor step in this evaluation which seems to have been overlooked by previous authors. This is probably because this observation is irrelevant for shapes with spherical topology, which is the case implicitly considered by previous authors, but becomes necessary when justifying this classical result for toroidal topologies. We then proceed to the main result in this paper, which is the evaluation of the general expression for the second variation near an equilibrium configuration (equation (3.9)). See also Sero-Guillaume (1983) for an analogous evaluation for ferromagnetic materials. This configuration will then be stable if this expression is strictly positive for all perturbations of the domain. After using this result to show the instability of some simple shapes, we investigate the effect of enclosing the whole system in a metal shield, which would then prevent the magnetic field from escaping. It turns out that the shield has a general stabilizing effect, whose exact nature depends on the topology of the liquid shape and on the field structure on its surface. We discuss the implications for various configurations.

Throughout this analysis we will consider both spherical and toroidal topologies; however, the results hold also in two dimensions, providing the currents induced in the liquid remain constant under variations of the shape. Finally, we assume that the external applied currents remain unchanged under perturbations of the liquid shape; this can be achieved in practice if there is a large impedance between the generator and the induction coils.

1.1. *Background theory and basic assumptions*

When a metal is placed in a single-phase field $\mathrm{Re}(\sqrt{2}\,\boldsymbol{B}(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega t})$, where $\boldsymbol{B}$ is its time r.m.s. value, eddy currents are induced which on interaction with the field produce a Lorentz force in the metal. The penetration of the field into the metal is governed by the induction equation:

$$\mathrm{i}\omega\boldsymbol{B} = \boldsymbol{\nabla} \wedge (\boldsymbol{u} \wedge \boldsymbol{B}) + \lambda\nabla^2\boldsymbol{B},$$

where $\boldsymbol{u}$ is the fluid velocity and λ the magnetic diffusivity. By increasing the frequency while keeping the field scale fixed, and assuming we can neglect the velocity term (meaning that the magnetic Reynolds number $R_m = U_0 L/\lambda \ll 1$, L being the length

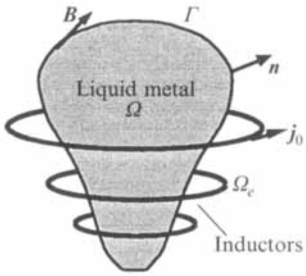

FIGURE 1. Definition of the problem.

scale of the liquid shape), the induction equation leads to an exponential decay of the field:

$$B \sim \exp[-x/\delta_m],$$

where x is the normal coordinate into the liquid domain and $\delta_m = (2\lambda/\omega)^{1/2}$ is the skin depth. Hence for sufficiently high frequencies the penetration of the field into the metal becomes negligible. In this limit the field exerts an inward normal (time-averaged) surface pressure $p_m = B^2/2\mu_0$ on the fluid.

The consistency of this approximation was verified by Mestel (1982) who showed that an upper estimate of the terminal fluid velocity is given by

$$U_0 \sim \frac{B_0^2\,\delta_m}{2\mu_0\,\rho\nu} \tag{1.1}$$

(ν being the viscosity). This not only justifies the initial assumption that $R_m \rightarrow 0$ with δ_m, but also shows that the inertia of the fluid becomes negligible compared with the magnetic pressure p_m. Hence in this limit the boundary condition on the stationary liquid surface is given by

$$\frac{B^2}{2\mu_0} + \rho g z + \gamma C = P, \tag{1.2}$$

where P is a constant and represents the jump in the modified pressure across the liquid boundary, γ is surface tension and C the mean curvature on the liquid surface. The r.m.s. field $\boldsymbol{B}$ is determined by the magnetostatic problem:

$$\nabla \wedge \boldsymbol{B} = \mu_0 \boldsymbol{j}_0, \nabla \cdot \boldsymbol{B} = 0 \quad \text{in} \quad \Omega_e, \tag{1.3a}$$

$$\boldsymbol{B} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma, \tag{1.3b}$$

where Γ is the liquid surface, Ω_e the region exterior to the metal, and $\boldsymbol{j}_0$ the r.m.s. of the external current distribution, assumed located in a system of inductor coils (figure 1).

It should be noted however that (1.1) assumes a small Reynolds number. In practice this may not the case for realistic parameter values: the Reynolds number: $Re = U_0 L/\nu \sim p_m \delta_m L/\rho\nu^2$ can be estimated from the requirement that the total magnetic levitating force ($\sim p_m L^2$) sustaining the liquid metal must balance at least the weight ($\sim \rho L^3 g$), so that $p_m > \rho L g$. Hence $Re \gtrsim \delta_m L^2 g/\nu^2$. For realistic physical situations, with $L \sim 10$ mm $\delta_m \sim 1$ mm, $\nu \sim 10^{-6}$ m^2 s^{-1}, we find that $Re \sim 10^6$! This flow is of course impossible and is partly due to the overestimate of (1.1) which assumes a steady laminar flow. But even so we must assume that the flow is turbulent for these typical parameter values. Then it is the Reynolds stresses that govern the flow in the boundary layer rather than the viscous stresses. In this case (1.1) is replaced by $U_0 \sim (gL)^{1/2}$ (Sneyd & Moffatt 1982), which gives the more reasonable value

$Re \sim 3000$. Unfortunately this implies that the dynamic pressure becomes of the same order as the electromagnetic pressure, thus invalidating the assumption that the internal fluid motion does not influence the shape of the liquid.

In spite of these observations we will neglect the internal fluid motion in all of our analysis. Although this is not always justified in practice, it can still provide us with an insight into the stability behaviour of these levitation devices.

2. The energy functional and its first variation

If we assume that the applied external currents remain constant under variations of the liquid domain Ω then the appropriate 'energy' functional of our system is given by Sneyd & Moffatt (1982):

$$E(\Omega) = -\int_{\Omega_e} \frac{B^2}{2\mu_0} \mathrm{d}V + \gamma \int_{\Gamma} \mathrm{d}S + \int_{\Omega} \rho g z \,\mathrm{d}V$$
$$\equiv E_m(\Omega) + E_\gamma(\Omega) + E_g(\Omega). \tag{2.1}$$

(Brancher 1980 also used $E(\Omega)$ in the analogous context of ferromagnetic liquids. See also Brancher & Sero-Guillaume 1983). Note the appearance of the minus sign in front of the magnetic contribution. This is accounted for by the fact that work is done in the external circuits to maintain currents of constant amplitude.

The first variation is a well known result:

$$\delta E = \int_{\Gamma} \left(\frac{B^2}{2\mu_0} + \gamma C + \rho g z \right) \boldsymbol{\tau} \cdot \boldsymbol{n} \,\mathrm{d}S. \tag{2.2}$$

2.1. *The first variation of the magnetic term*

In this section we will re-derive the variation of the magnetic term E_m, paying attention to spherical, two-dimensional and toroidal topologies for the liquid shape.

Following Sero Guillaume's approach, this can be derived by deforming the domain Ω by a transformation on R^3 depending on a parameter ϵ:

$$\boldsymbol{x}_\epsilon = \boldsymbol{T}_\epsilon(\boldsymbol{x}_0),$$

where $\boldsymbol{T}_\epsilon = \boldsymbol{I}$ (the identity) at $\epsilon = 0$ and $\boldsymbol{T}_\epsilon$ is differentiable around $\epsilon = 0$. The transformation $\boldsymbol{T}_\epsilon$ is a flow field in R^3, so that the points in R^3 can be thought of as fluid particles moving as the 'time' parameter ϵ varies, thereby deforming the boundary Γ. The 'rate of flow' of these points is given by

$$\boldsymbol{\tau}_\epsilon(\boldsymbol{x}_\epsilon) \equiv \frac{\mathrm{d}\boldsymbol{T}_\epsilon(\boldsymbol{x}_0)}{\mathrm{d}\epsilon} \quad \text{with} \quad \boldsymbol{x}_0 = \boldsymbol{T}_\epsilon^{-1}(\boldsymbol{x}_\epsilon),$$

where $\boldsymbol{x}_\epsilon$ are coordinates of the point $\boldsymbol{x}_0$ transported under $\boldsymbol{T}_\epsilon$. Hence $\boldsymbol{\tau}_\epsilon(\boldsymbol{x}_\epsilon)$ is just the Eulerian velocity field of the points in R^3.

The variation δE of E with respect to $\boldsymbol{T}_\epsilon$ is then simply the derivative of $E(\boldsymbol{T}_\epsilon(\Omega))$ with respect to ϵ at $\epsilon = 0$. The analogy with Eulerian kinematics is now complete, so to derive the magnetic contribution we use the classical formula for the total derivative:

$$\delta \int_{\Omega_e} \tfrac{1}{2} B^2 \,\mathrm{d}V = \int_{\Omega_e} \left[\boldsymbol{B} \cdot \mathrm{D}\boldsymbol{B} + \tfrac{1}{2} B^2 \boldsymbol{\nabla} \cdot \boldsymbol{\tau} \right] \mathrm{d}V, \tag{2.3}$$

where $$\mathrm{D}\boldsymbol{B} \equiv \left. \frac{\mathrm{D}\boldsymbol{B}_\epsilon(\boldsymbol{x}_\epsilon)}{\mathrm{D}\epsilon} \right|_{\epsilon=0},$$

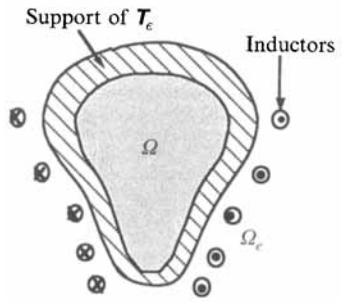

FIGURE 2. Deformation of the external domain Ω_e by T_ϵ.

the familiar transported derivative. The variation $\delta \boldsymbol{B}$ is given by

$$\delta \boldsymbol{B} \equiv \mathrm{D}\boldsymbol{B} - (\boldsymbol{\tau}\cdot\nabla)\boldsymbol{B}. \tag{2.4}$$

Which is the change of $\boldsymbol{B}$ due only to the deformation of the liquid domain

$$\delta \boldsymbol{B}(\boldsymbol{x}) = \left.\frac{\partial \boldsymbol{B}_\epsilon(\boldsymbol{x})}{\partial \epsilon}\right|_{\epsilon=0}. \tag{2.5}$$

note that the above indirect definition (2.4) of $\delta \boldsymbol{B}$ avoids problems caused by $\boldsymbol{B}_\epsilon$ not being well defined at $\epsilon = 0$, since $\boldsymbol{B}$ is discontinuous across the moving liquid boundary. From (2.5) it is obvious that δ commutes with the derivative operator:

$$\frac{\partial \delta \boldsymbol{B}}{\partial x^i} = \delta \frac{\partial \boldsymbol{B}}{\partial x^i}$$

(the same is of course not true for the operator D). This definition of the operator δ is analogously extended to any domain-dependent function. Using (2.4) and (2.3) and integrating by parts, we obtain

$$\delta E_m = \frac{1}{\mu_0}\int_{\Gamma - S_\infty} \tfrac{1}{2}B^2 \boldsymbol{\tau}\cdot\boldsymbol{n}\,\mathrm{d}S - \frac{1}{\mu_0}\boldsymbol{B}\cdot\delta\boldsymbol{B}\,\mathrm{d}V,$$

where S_∞ is an 'outer' surface at infinity (note that n is the outward normal).

At this point we note that the transformation $\boldsymbol{T}_\epsilon$ also deforms the external domain Ω_e. Without loss of generality we only consider transformations which reduce to the identity everywhere in Ω_e, except in a neighbourhood of the liquid surface excluding the surrounding inductors (figure 2). The contribution of the 'outer' surfaces S_∞ therefore vanishes because $\boldsymbol{\tau}$ vanishes at a finite distance from Ω.

The second term also vanishes, but the argument differs slightly for different topologies. We first observe that in Ω_e, $\nabla \wedge \delta\boldsymbol{B} = \delta(\nabla \wedge \boldsymbol{B}) = \mu_0 \delta \boldsymbol{j}_0 = 0$, as we assume that the applied currents remain constant in time. Hence $\delta \boldsymbol{B} = \nabla\varphi$ for some function φ. Here we must distinguish between different topologies.

The simplest case is if Γ has spherical topology. Then φ will always be single valued, since we have $\int \delta\boldsymbol{B}\cdot\mathbf{d}\boldsymbol{l} = 0$ for any loop in Ω_e. Hence the second term is

$$\int_{\Omega_e} \boldsymbol{B}\cdot\delta\boldsymbol{B}\,\mathrm{d}V = \int_{\Omega_e} \boldsymbol{B}\cdot\nabla\varphi\,\mathrm{d}V = \int_{\Gamma+S_\infty} \varphi\boldsymbol{B}\cdot\boldsymbol{n}\,\mathrm{d}S$$

on integration by parts (S_∞ is a surface at infinity). But $\boldsymbol{B}\cdot\boldsymbol{n} = 0$ on Γ, so this integral vanishes. The contribution at infinity vanishes since $\boldsymbol{B}$ (and $\delta\boldsymbol{B}$) has a dipole behaviour at large distances.

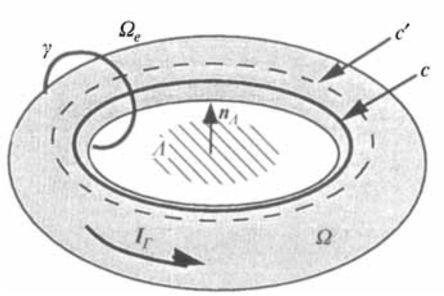

FIGURE 3. The surface of discontinuity A for the field potential, chosen as spanning the surface curve c. The adjacent curve c' runs just inside the liquid shape Ω.

In the two-dimensional on the other hand, the potential φ will generally be multivalued, since it presents a discontinuity equal to $\oint_\gamma \delta \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{l} = \mu_0\, \delta I_\Gamma$ (where γ is any loop surrounding the liquid shape). Hence in order for the above analysis to apply we *must assume that the total current I_Γ flowing in Ω remains constant as Ω varies* (i.e. $\delta I_\Gamma = 0$). Some examples in which this can occur will be given later.

We would think that the same condition should apply to toroidal shapes, since in general the total induced current I_Γ flowing in the torus also varies under surface variations. However this condition turns out to be redundant: the potential φ will generally present a surface of discontinuity A in Ω_e, chosen as spanning a curve c on Γ (figure 3), over which φ jumps by the quantity $\oint_\gamma \delta\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{l} = \mu_0\,\delta I_\Gamma$. Hence on integration by parts we therefore end up with the additional term $\mu_0\,\delta I_\Gamma \int_A \boldsymbol{B}\cdot\boldsymbol{n}_A\,\mathrm{d}S$, $\boldsymbol{n}_A$ being the normal to A. However this integral turns out to be zero because of the induction equation:

$$\mathrm{i}\omega \boldsymbol{B} = -\boldsymbol{\nabla}\wedge \boldsymbol{E}.$$

In fact, on integration over a surface A' spanning the curve c' running just inside the liquid domain Ω, we obtain

$$\mathrm{i}\omega \int_{A'} \boldsymbol{B}\cdot\boldsymbol{n}_A\,\mathrm{d}S = -\int_{c'} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} = 0$$

since $\boldsymbol{E}$ is zero inside Ω (see the Appendix for details).

But the field $\boldsymbol{B}$ remains bounded on Γ, so that the surface integral varies continuously as c' crosses the boundary Γ. We therefore must have the consistency condition:

$$\int_A \boldsymbol{B}\cdot\boldsymbol{n}_A\,\mathrm{d}S = 0. \tag{2.6}$$

We conclude that the term $\int_{\Omega_e} \boldsymbol{B}\cdot\delta\boldsymbol{B}\,\mathrm{d}V$ in the expression for δE_m vanishes in all cases, except in two dimensions, unless the current flowing through the shape remains constant under shape variations, so that

$$\delta E_m = \frac{1}{\mu_0}\int_\Gamma \tfrac{1}{2}B^2\,\boldsymbol{\tau}\cdot\boldsymbol{n}\,\mathrm{d}S. \tag{2.7}$$

2.2. *The variational principle*

Consider variations of E (2.1) under deformations preserving the volume $|\Omega|$ of Ω, implying that $\boldsymbol{\tau}$ must satisfy

$$\delta|\Omega| = \int_\Gamma \boldsymbol{\tau}\cdot\boldsymbol{n}\,\mathrm{d}S = 0. \tag{2.8}$$

The equilibrium equation (1.2) is then recovered by the variational statement:

$$\delta E - P\,\delta|\Omega| = 0 \quad \text{for any} \quad \tau,$$

where P appears as a Lagrange multiplier for the constraint (2.8) on the deformation field. Therefore the domain Ω satisfying the equilibrium boundary condition is also a stationary 'point' for the energy E in the space of all volume-preserving deformations.

3. Stability and the second variation of the energy

3.1. *The second variation*

The stable solutions correspond to the domains Ω for which E is a minimum. Let Ω be such a stationary domain. Then for any deformation T_ϵ:

$$E(T_\epsilon(\Omega)) = E(\Omega) + \tfrac{1}{2}\epsilon^2\delta^2 E(\Omega) + \dots \quad \text{with} \quad \delta^n \equiv \mathrm{d}^n/\mathrm{d}\epsilon^n|_{\epsilon=0}.$$

The functional E will then be at a minimum if $\delta^2 E(\Omega)$ is strictly positive for all deformations preserving the volume of Ω. (Note that if $\delta^2 E(\Omega)$ is zero for any particular deformation T we would then have a look at higher derivatives of E in order to examine the nature of this stationary point.)

One could ask if this is a sufficient condition for stability, that is all stationary domains with non-minimal E correspond to unstable configurations, as one could envisage Arnold type stabilities corresponding to a maximum energy. However, these situations are probably ruled out by dissipation due to viscous effects, which lead to a uniform decrease in the total energy whenever the system is in movement, which should therefore tend to relax to a state with lower E.

In order to compute $\delta^2 E(\Omega)$ we must recover the expression for $\mathrm{d}E/\mathrm{d}\epsilon$. This is simply done because (2.2) is in fact the general expression for the rate of change of E. After re-introducing the ϵ dependence in (2.2), differentiating and setting ϵ to zero we obtain

$$\delta^2 E = \int_\Gamma \mathrm{D}[F\boldsymbol{\tau}\cdot\boldsymbol{n}J]\,\mathrm{d}S = \int_\Gamma \mathrm{D}F\boldsymbol{\tau}\cdot\boldsymbol{n}\,\mathrm{d}S + \int_\Gamma F\mathrm{D}[\boldsymbol{\tau}\cdot\boldsymbol{n}J]\,\mathrm{d}S,$$

with

$$F(\boldsymbol{x}) = \frac{B^2(\boldsymbol{x})}{2\mu_0} + \gamma C(\boldsymbol{x}) + \rho g z(\boldsymbol{x}).$$

J stands for $J_\epsilon(\boldsymbol{x}_\epsilon)$, the surface Jacobian resulting from the variation of the surface coordinates, and following our conventions we have $J_0(\boldsymbol{x}_0) = 1$. Since we only require $\delta^2 E$ for an equilibrium configuration, we can use the stationary condition (1.2) to rewrite the second term:

$$\int_\Gamma F\mathrm{D}[\boldsymbol{\tau}\cdot\boldsymbol{n}J]\,\mathrm{d}S = P\int_\Gamma \mathrm{D}[\boldsymbol{\tau}\cdot\boldsymbol{n}J]\,\mathrm{d}S.$$

However we are only considering perturbations preserving the volume $|\Omega|$ of the liquid, so that $\delta^n|\Omega| = 0$ for all n. Hence by differentiating (2.8) using the same procedure we obtain

$$\delta^2|\Omega| = \int_\Gamma \mathrm{D}[\boldsymbol{\tau}\cdot\boldsymbol{n}J]\,\mathrm{d}S = 0.$$

So the second term in $\delta^2 E$ vanishes. Hence for an equilibrium configuration we are left with

$$\delta^2 E = \int_\Gamma \mathrm{D}F\boldsymbol{\tau}\cdot\boldsymbol{n}\,\mathrm{d}S \quad \text{with} \quad \mathrm{D}F = \mathrm{D}\frac{B^2}{2\mu_0} + \gamma\mathrm{D}C + \rho g\,\mathrm{D}z. \tag{3.1}$$

δF is just the change in the total pressure due to the surface deformation.

3.2. *Evaluation of* $\mathrm{D}\frac{1}{2}B^2$

From now on we will assume that the surface Γ is smooth, so that $\boldsymbol{B}$ will be smooth right up to γ. Hence we can write

$$\mathrm{D}\tfrac{1}{2}B^2 = \boldsymbol{\tau}\cdot\nabla\tfrac{1}{2}B^2+\delta\tfrac{1}{2}B^2 \quad \text{on} \quad \Gamma.$$

In the calculations that follow, we will simplify the expressions by using the components parallel and perpendicular to Γ. We will also take the normal $\boldsymbol{n}$ as being the function defined in an open neighbourhood in R^3 of Γ by

$$\boldsymbol{n} = \nabla\Psi/|\nabla\Psi|,$$

where $\Psi(x,^1, x^2, x^3) = 0$ is a local representation of Γ. For example we can choose Ψ so that $\boldsymbol{n}$ is the outward normal. Now let $\boldsymbol{d}$ be defined by

$$\boldsymbol{d} \equiv \nabla - \boldsymbol{n}(\boldsymbol{n}\cdot\nabla),$$

so that $\boldsymbol{n}\cdot\boldsymbol{d} \equiv 0$. Note that $\boldsymbol{d}$ is defined in the same open n.h.d. in R^3 as $\boldsymbol{n}$, and on Γ, $\boldsymbol{d}$ is just the surface derivative ∇_s. Hence on Γ

$$\mathrm{D}\tfrac{1}{2}B^2 = \boldsymbol{\tau}_t\cdot\nabla_s\tfrac{1}{2}B^2+\tau_n\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial n}+\boldsymbol{B}\cdot\delta\boldsymbol{B},$$

where τ_t, τ_n are respectively the tangential and normal components of τ. Note that only normal displacements really need to be considered, as τ_t does not deform the liquid surface. We will therefore neglect τ_t in what follows. The scrupulous reader may wish to verify that the τ_t terms would have in any case disappeared in the calculations.

We can rewrite the second term using the fact that $\boldsymbol{B}$ is irrotational in a one-sided neighbourhood of Γ, so that the Cartesian components of $\boldsymbol{B}$ satisfy

$$\partial_j B^i = \partial_i B^j \quad \text{where} \quad \partial_i \equiv \partial/\partial x^j.$$

Hence on Γ we have since $B^i\partial_i = B^i\mathrm{d}_i$,

$$\begin{aligned}\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial n} &= B^i n^j\partial_j B^i = B^i n^j \mathrm{d}_i B^j\\ &= B^i\mathrm{d}_i(\boldsymbol{n}\cdot\boldsymbol{B}) - B^i B^j \mathrm{d}_i n^j.\end{aligned}$$

The first term vanishes since $\boldsymbol{n}\cdot\boldsymbol{B}=0$ and the tensor $\boldsymbol{dn}$ is simply the second fundamental form π, so

$$B^i B^j \mathrm{d}_i n^j \equiv \pi(\boldsymbol{B},\boldsymbol{B}) = \kappa_B|\boldsymbol{B}|^2, \tag{3.2}$$

where κ_B is the normal surface curvature along the field line $\boldsymbol{B}$. Note that κ_B is always positive for convex surfaces, as we take $\boldsymbol{n}$ as being the outward normal. Hence on substitution we obtain

$$\mathrm{D}\tfrac{1}{2}B^2 = -\tau_n\kappa_B|\boldsymbol{B}|^2+\boldsymbol{B}\cdot\delta\boldsymbol{B}. \tag{3.3}$$

Note that the first term is the contribution arising from the change in the field on the surface due only to surface displacements, while the second one arises from the field deformation. We will discuss the effect of each term later in §3.5.

3.3. *Equations for* $\delta\boldsymbol{B}$

We now show that the equations satisfied by $\delta\boldsymbol{B}$ are

$$\nabla\cdot\delta\boldsymbol{B} = \nabla\wedge\delta\boldsymbol{B} = 0 \quad \text{in} \quad \Omega_e, \tag{3.4a}$$

$$\boldsymbol{n}\cdot\delta\boldsymbol{B} = \nabla_s\cdot(\tau_n\boldsymbol{B}) \quad \text{on} \quad \Gamma. \tag{3.4b}$$

Equations (3.4*a*) are immediately obtained from (1.3*a*) by the commutativity of δ and ∇, and the assumption that the external currents remain constant, so that $\delta \boldsymbol{j} = 0$. Equation (3.4*b*) follows from the fact that $\boldsymbol{n}\cdot\boldsymbol{B}$ remains zero on the moving boundary:

$$\mathrm{D}(\boldsymbol{n}\cdot\boldsymbol{B}) = 0.$$

This implies that

$$0 = \mathrm{D}\boldsymbol{n}\cdot\boldsymbol{B} + \boldsymbol{n}\cdot\mathrm{D}\boldsymbol{B}$$

$$= \mathrm{D}\boldsymbol{n}\cdot\boldsymbol{B} + \boldsymbol{n}\cdot(\boldsymbol{\tau}\cdot\nabla)\,\boldsymbol{B} + \boldsymbol{n}\cdot\delta\boldsymbol{B}.$$

So

$$\boldsymbol{n}\cdot\delta\boldsymbol{B} = -\mathrm{D}\boldsymbol{n}\cdot\boldsymbol{B} - \tau_n\,\boldsymbol{n}\cdot\partial\boldsymbol{B}/\partial n.$$

Rewrite the last term using the identity

$$0 = \nabla\cdot\boldsymbol{B} = \nabla_s\cdot\boldsymbol{B} + \boldsymbol{n}\cdot\partial\boldsymbol{B}/\partial n,$$

where $\nabla_s\cdot\boldsymbol{B}$ is the surface divergence of $\boldsymbol{B}$:

$$\boldsymbol{n}\cdot\delta\boldsymbol{B} = -\boldsymbol{B}\cdot\mathrm{D}\boldsymbol{n} + \tau_n\,\nabla_s\cdot\boldsymbol{B}.$$

Now we know that, for normal displacements,

$$\mathrm{D}\boldsymbol{n} = -\nabla_s\,\tau_n$$

(see the Appendix for proof). Hence

$$\boldsymbol{n}\cdot\delta\boldsymbol{B} = \boldsymbol{B}\cdot\nabla_s\,\tau_n + \tau_n\,\nabla_s\cdot\boldsymbol{B} = \nabla_s\cdot(\tau_n\,\boldsymbol{B}).$$

Finally note that for spherical topologies, $\delta\boldsymbol{B} = \nabla\varphi$ for a single-valued φ. These equations then become

$$\nabla^2\varphi = 0 \quad \text{in} \quad \Omega_e, \tag{3.5a}$$

$$\partial\varphi/\partial n = \nabla_s\cdot(\tau_n\,\boldsymbol{B}) \quad \text{on} \quad \Gamma, \tag{3.5b}$$

$$\varphi \quad \text{bounded at infinity.} \tag{3.5c}$$

These equations also apply to two-dimensional shapes for which the total current through the cross-section remains unchanged: $\delta I_\Gamma = 0$.

3.4. *Evaluation of* $\delta^2 E$

We know (see the Appendix) that in the case of normal displacements ($\tau_t = 0$) the variation of C (defined here as the sum of the principal curvatures, and not of their average) is given by

$$\mathrm{D}C = -(\nabla_s^2\,\tau_n + \alpha^2\tau_n), \tag{3.6}$$

where $\nabla_s^2 \equiv \mathrm{d}_i\,\mathrm{d}_i$ is the surface Laplacian (Beltrami) operator and

$$\alpha^2 = \kappa_1^2 + \kappa_2^2,$$

κ_1, κ_2 being the principal curvatures. We also have

$$\mathrm{D}z = \boldsymbol{\tau}\cdot\nabla z = \tau_n\,\hat{\boldsymbol{z}}\cdot\boldsymbol{n}. \tag{3.7}$$

Substituting (3.3), (3.6), (3.7) into DF given in (3.1) we obtain

$$\delta F = -\frac{1}{\mu}\tau_n\,\kappa_B\,|\boldsymbol{B}|^2 + \frac{1}{\mu_0}\boldsymbol{B}\cdot\delta\boldsymbol{B} - \gamma(\nabla_s^2\,\tau_n + \alpha^2\tau_n) + \rho g\hat{\boldsymbol{z}}\cdot\boldsymbol{n}\tau_n. \tag{3.8}$$

Therefore substituting (3.8) into (3.1) and noting that

$$\int_\Gamma \tau_n \nabla_s^2 \tau_n \,\mathrm{d}S = \int_\Gamma [\nabla_s\cdot(\tau_n \nabla_s \tau_n) - (\nabla_s \tau_n)\cdot(\nabla_s \tau_n)]\,\mathrm{d}S = -\int_\Gamma |\nabla_s \tau_n|^2 \,\mathrm{d}S$$

(the first term vanishes by integration by parts, since Γ is a closed surface) we obtain

$$\delta^2 E = \int_\Gamma \left[-\frac{1}{\mu_0}\kappa_B \tau_n^2 |\boldsymbol{B}|^2 + \frac{1}{\mu_0}\tau_n \boldsymbol{B}\cdot\delta\boldsymbol{B} + \gamma(|\nabla_s \tau_n|^2 - \alpha^2\tau_n^2) + \rho g \hat{\boldsymbol{z}}\cdot\boldsymbol{n}\tau_n^2\right]\mathrm{d}S, \quad (3.9)$$

where $\boldsymbol{B}$ satisfies the equilibrium equation (1.2) and $\delta\boldsymbol{B}$ is the solution of (3.4). We can write

$$\delta^2 E = \delta^2 E_m + \delta^2 E_\gamma + \delta^2 E_g,$$

with

$$\delta^2 E_m = \frac{1}{\mu_0}\int_\Gamma [-\kappa_B \tau_n^2 |\boldsymbol{B}|^2 + \tau_n \boldsymbol{B}\cdot\delta\boldsymbol{B}]\,\mathrm{d}S, \quad (3.10a)$$

$$\delta^2 E_\gamma = \gamma\int_\Gamma (|\nabla_s \tau_n|^2 - \alpha^2\tau_n^2)\,\mathrm{d}S; \quad \delta^2 E_g = \rho g \int_\Gamma \hat{\boldsymbol{z}}\cdot\boldsymbol{n}\tau_n^2\,\mathrm{d}S, \quad (3.10b,c)$$

which are respectively the magnetic, surface tension and gravitational energy contributions. Finally remember that τ_n must satisfy the volume-preserving condition: $\int_\Gamma \tau_n \,\mathrm{d}S = 0$.

3.5. *Discussion of the magnetic term*

The second term in $\delta^2 E_m$ (3.10*a*):

$$\frac{1}{\mu_0}\int_\Gamma \tau_n \boldsymbol{B}\cdot\delta\boldsymbol{B}\,\mathrm{d}S,$$

is the contribution due purely to the change in the magnetic field surrounding the liquid, because

$$\int_\Gamma \tau_n \boldsymbol{B}\cdot\delta\boldsymbol{B}\,\mathrm{d}S = \int_{\Omega_e} |\delta\boldsymbol{B}|^2\,\mathrm{d}V. \quad (3.11)$$

In fact, for spherical and two-dimensional topologies

$$\int_\Gamma \tau_n \boldsymbol{B}\cdot\delta\boldsymbol{B}\,\mathrm{d}S = \int_\Gamma \tau_n \boldsymbol{B}\cdot\nabla_s\varphi\,\mathrm{d}S$$

$$= \int_\Gamma [\nabla_s\cdot(\tau_n \boldsymbol{B}\varphi) - \varphi\nabla_s\cdot(\tau_n \boldsymbol{B})]\,\mathrm{d}S.$$

The first term vanishes by integration, since Γ is a closed surface. So using (3.4*b*)

$$\int_\Gamma \boldsymbol{B}\cdot\delta\boldsymbol{B}\tau_n\,\mathrm{d}S = -\int_\Gamma \varphi\delta\boldsymbol{B}\cdot\boldsymbol{n}\,\mathrm{d}S$$

which using the divergence theorem,

$$= \int_{\Omega_e} |\delta\boldsymbol{B}|^2\,\mathrm{d}V$$

Note in particular that this term is always positive, and therefore has a stabilizing effect. Physically this term is the contribution arising from the repelling effect due to the compression of the magnetic field. We therefore refer to (3.11) as the *compression term.*

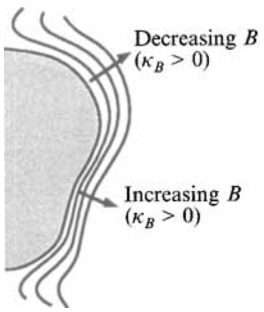

FIGURE 4. The integrand of the curvature term will be destabilizing in convex regions of the liquid surface (where $\kappa_B > 0$), and stabilizing in concave regions (where $\kappa_B < 0$).

The expression (3.11) is also true for toroidal shapes, although the following extra terms appear when integrating by parts, owing to the jump in φ:

$$\delta I_\Gamma \int_A \delta \boldsymbol{B}\cdot\boldsymbol{n}_A \,\mathrm{d}S + \delta I_\Gamma \int_\Gamma \boldsymbol{B}\cdot\boldsymbol{n}_A \tau_n \,\mathrm{d}l$$

(where A is the surface in figure 3). But this is just

$$\delta I_\Gamma \delta \int_A \boldsymbol{B}\cdot\boldsymbol{n}_A \,\mathrm{d}S$$

so this integral vanishes using (2.6).

On the other hand, the first term in $\delta^2 E_m$ (3.10*a*):

$$\frac{1}{\mu_0}\int_\Gamma -\kappa_B \tau_n^2 B^2 \,\mathrm{d}S,$$

(we will call this the *curvature term*) is the contribution arising from the change in the field on the surface due only to surface displacements (and not the field deformation), as the integrand is just $\tau_n^2(\partial/\partial n)\frac{1}{2}B^2$ (see §3.2), where $\tau_n(\partial/\partial n)\frac{1}{2}B^2$ corresponds to the change in the magnetic pressure along the virtual displacement τ_n. The integrand is therefore positive if the magnetic field increases away from the surface, having the physical interpretation that an increase in the magnetic pressure under a positive virtual displacement provides a stabilizing restoring force.

The sign of the integrand is given by κ_B, the normal curvature along a surface field line. An interesting geometrical interpretation can be derived from the behaviour of this term: if $|s|$ is the length of any infinitesimal segment on the surface, then its change due to a normal surface deformation is given by $\kappa_s|s|\tau_n$ where κ_s is the normal curvature along s. Hence the integrand is negative if the surface field lines are *stretched* for positive (i.e. outer) τ_n. This of course corresponds to regions in which the magnetic field decreases away from the surface, since $\partial|\boldsymbol{B}|/\partial n = -\kappa_B|\boldsymbol{B}|$. Note that in any case the integrand is always negative in any convex region on the surface (figure 4), whatever the shape of the surface field. In particular *the curvature term will always be negative (and therefore destabilizing) for totally convex surfaces*, such as the sphere or any slightly perturbed sphere (or the circle in the two-dimensional case).

Therefore for stability, the (possibly negative) curvature term must be outweighed by the (positive) compression term. The above observation indicates that the 'negative influence' of the curvature term is attenuated for shapes with concave regions, such as the cruciform shapes calculated by Shercliff (1981), or the 'upside-down pair shapes' so familiar in axisymmetric levitation devices, both of which are known to be stable:

here the curvature term is 'sufficiently non-negative' to be outweighed by the positive compression term.

Finally note that only the compression term depends on the surface derivative of τ_n. In fact the magnitude of $\delta\boldsymbol{B}$ is determined by its boundary condition (3.4*b*) which can be rewritten as

$$\boldsymbol{n}\cdot\delta\boldsymbol{B} = \boldsymbol{B}\cdot\nabla_s\tau_n + \tau_n\nabla_s\cdot\boldsymbol{B}.$$

This indicates that the compression term becomes relatively large for perturbations which are sufficiently 'rippled' in the direction of the field on the surface (i.e. for large $\boldsymbol{B}\cdot\nabla_s\tau_n$). We therefore recover the familiar result that the magnetic field stabilizes high-frequency disturbances along the field lines.

In conclusion, the above observations indicate that the least-stable configurations arise with totally convex surfaces, particularly when subjected to 'low-frequency' deformations. This observation is confirmed in the examples below.

4. An example in two dimensions

We will now analyse the stability of a horizontally levitated cylinder with uniform circular cross-section (figure 5). We assume that a current distribution necessary for obtaining this shape has been determined in a set of coils surrounding the liquid in the form of a concentric cylinder. This is done by solving the associated inverse problem. We will not dwell on the details of this calculation, as these currents do not appear in the expression (3.9) for $\delta^2 E$ (for the solution to this inverse problem refer to Felici 1991), but it is necessary to emphasize that this stability analysis is valid for the two-dimensional case only if the total current induced in the liquid remains constant as the shape varies. This can be achieved if, for example, the liquid cylinder is somehow connected at the endpoints with the external coils (via a three-dimensional circuit), thereby forcing the total induced current I_Γ to be equal to the total current in the wires I_S, also constant if we assume a large impedance between the coils and the generator.

A more realistic situation would be to consider the cylinder as being a section of a torus with a radius R_τ large compared to its cross-section. In this case condition (2.6) reduces to

$$0 = \int_A \boldsymbol{B}\cdot\boldsymbol{n}_A \,\mathrm{d}S$$
$$\sim (I_\Gamma + I_S)\, R_\tau \ln R_\tau$$

to leading order as $R_\tau \to \infty$ (see the Appendix). Hence in this limit we must have $I_\Gamma = -I_S$ (this is a generalization of Sneyd & Moffatt's result who showed the equality for circular cross-sections by considering the mutual induction of two circles).

Finally we notice that these shapes might be subject to longitudinal instabilities, which are excluded from this analysis. We will discuss these later.

Under these assumptions, we will calculate $\delta^2 E$ for rigid (cross-sectional) displacements, and show that this is an unstable configuration. See Descloux (1991) for an analogous result for any two-dimensional configuration, but in the absence of gravity and surface tension. For rigid displacements, $\delta^2 E_C$ and $\delta^2 E_g$ both vanish so the only contribution left is the magnetic term. In the two-dimensional case the surface will only have one radius of curvature C and the surface derivative operator ∇_s reduces to $\mathrm{d}/\mathrm{d}s$ (s is the line distance on Γ). Hence (3.9) becomes

$$\delta^2 E = \frac{1}{\mu_0}\int_\Gamma \left(-\tau_n^2 B^2 C - \tau_n \varphi \frac{\partial\varphi}{\partial n}\right)\mathrm{d}s,$$

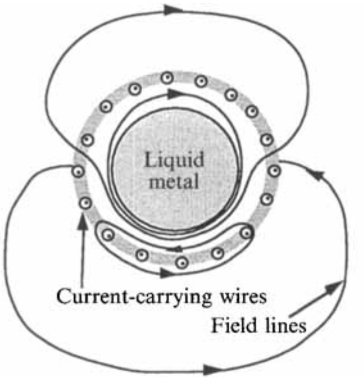

FIGURE 5. Levitation of horizontal liquid metal cylinder of circular cross-section using current inductors suitably distributed around an external concentric cylinder.

where s is the distance element on Γ. The second (compression) term has been rewritten using the potential function φ defined in (3.5). Rigid displacements of a circle are given in polar coordinates by

$$\tau_n = h_v \sin\theta + h_r \cos\theta;$$

h_v, h_r are constants which denote respectively the vertical and horizontal displacements. From the equilibrium equation (1.2)

$$B|_\Gamma = (\mu_0 \beta)^{1/2}(1-\alpha\sin\theta)^{1/2} \quad \text{with} \quad \beta = 2\left(P-\frac{\gamma}{R}\right), \quad \alpha = \rho g R\left(P-\frac{\gamma}{R}\right)^{-1},$$

where R is the radius of the liquid shape. As the curvature is constant ($C = 1/R$) the first term in $\delta^2 E$ integrates to

$$-\beta\pi(h_v^2+h_r^2).$$

The second (compression) term can be rewritten as

$$-\beta h_v^2 \int_0^{2\pi} \varsigma \frac{dG}{d\theta} d\theta,$$

where $G(\theta) = (1-\alpha\sin\theta)^{1/2}$ $(h_v \sin\theta + h_r\cos\theta)$, and the function ς is defined by $\varphi(rR, \theta) = (\mu_0\beta)^{1/2}\varsigma(r,\theta)$ on the rescaled domain $r > 1$, so that from (3.5)

$$\Delta\varsigma = 0 \quad \text{for} \quad r > 1, \tag{4.1a}$$

$$\frac{\partial \varsigma}{\partial r} = \frac{dG}{d\theta} \quad \text{on} \quad r = 1, \tag{4.1b}$$

$$\varsigma \text{ bounded at infinity.} \tag{4.1c}$$

We can solve (4.1) by expressing G in its Fourier series form:

$$G = G_0 + \sum_1^\infty (G_n e^{in\theta} + \bar{G}_n e^{in\theta}).$$

Substituting the result in the above integral and integrating term by term we obtain

$$\delta^2 E = -\beta\pi(h_v^2+h_r^2) + 4\pi\beta h_v^2 \sum_1^\infty n|G_n|^2. \tag{4.2}$$

We now make the simplifying assumption that the surface magnetic pressure is large compared with the gravitational pressure, so that we can take $\alpha(\equiv \rho g R \beta^{-1}) \ll 1$. This assumption is valid if the total applied current I_S is strong enough, because

$$I_S = -I_\Gamma = \frac{1}{2\pi\mu_0}\int_0^{2\pi} B_\Gamma(\theta)\,\mathrm{d}\theta = o(\beta)^{1/2}.$$

Then

$$G = (1-\tfrac{1}{2}\alpha\sin\theta - \tfrac{1}{8}\alpha^2\sin^2\theta)(h_v\sin\theta + h_r\cos\theta) + O(\alpha^3)$$

so that to this order

$$G_1 = \left(\frac{1}{2}-\frac{\alpha^2}{64}\right)h_r + \mathrm{i}\left(\frac{1}{2}-\frac{3\alpha^2}{64}\right)h_v, \quad G_2 = \frac{\alpha}{8}(h_v+\mathrm{i}h_r), \quad G_3 = \frac{\alpha^2(h_r-\mathrm{i}h_v)}{64}.$$

Substituting in (4.2) the relevant terms we finally obtain

$$\delta^2 E = \frac{\pi\beta\alpha^2}{16}(h_r^2 - h_v^2) + \dots.$$

The circular shape is therefore unstable under vertical displacements and stable under horizontal displacements.

This result remained unchanged even when dropping the assumption of small α. In this case the G_n had to evaluated numerically and the series (4.2) truncated for sufficiently large n. We therefore conclude that this shape is unstable regardless of our choice of inductor currents necessary for forming this shape.

5. An example in three dimensions

We now consider an almost spherical axisymmetric drop, thus making a perturbation analysis possible. We assume that Γ is given in spherical polars (r, θ, ϕ) by

$$r_\Gamma(\theta) = R(1+\epsilon\eta(\cos\theta)),$$

where $\eta(x)$ is any analytic function in the range $x \in [-1, 1]$, and is chosen in such a way that Γ is an admissible surface (i.e. surfaces that can be formed, for which therefore (1.2) admits a surface field $\boldsymbol{B}$).

The field lines of $\boldsymbol{B}$ lie along the lines of symmetry, and the magnitude is given by the equilibrium equation (1.2), which to leading order becomes

$$\frac{B^2}{2\mu_0} + \rho g R[1+\epsilon\eta(\cos\theta)]\cos\theta + \frac{\gamma}{R}(1-\epsilon C'(\cos\theta)) = P,$$

with

$$C' = \frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}\eta}{\mathrm{d}x}\right] + 2\eta; \quad x = \cos\theta,$$

where we have used (3.6) to write the perturbation C' for C. We rewrite the above equation in the non-dimensional form

$$b^2 + W[1+\epsilon\eta(\cos\theta)]\cos\theta - \epsilon C'(\cos\theta) = P,$$

where $b = (R/2\mu_0\gamma)^{1/2}B$, $W = \rho g R^2/\gamma$, and where we have incorporated all constant terms in the (rescaled) constant P. The latter is determined by the requirements that the field must vanish at the poles $\theta = 0, \pi$. These two conditions also provide a restriction on the possible surfaces, since we must have

$$P = W[1+\epsilon\eta(1)] - \epsilon C'(1) = -W[1+\epsilon\eta(-1)] - \epsilon C'(-1).$$

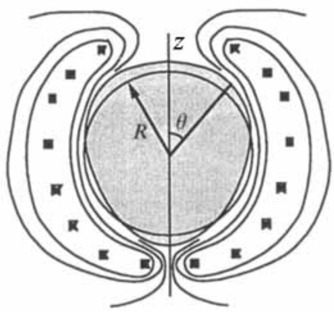

FIGURE 6. An axisymmetric levitation configuration with almost spherical liquid shape.

We deduce that the non-dimensional constant W must be of order ϵ. Therefore quasi-spherical surfaces can only be formed if the liquid drop is sufficiently small, so as to let surface tension become dominant over the weight of the drop. For aluminium, for example, taking $\epsilon = 0.1$ as a typically small value, we have that $R \approx 2$ mm! Moreover, the underlying assumption (among others) that the electromagnetic skin depth remains negligible ($\delta_m/R \ll 1$) becomes tenuous. The perturbation analysis therefore applies to these relatively unrealistic cases, but it nevertheless gives an indication of the stability for more general shapes.

To leading order in ϵ we therefore have

$$b^2 = P - C'(\cos\theta) - W'\cos\theta, \tag{5.1}$$

with $W = \epsilon W'$ (hence $W' = O(1)$) and

$$P = \tfrac{1}{2}[C'(1) + C'(-1)]; \quad W' = \tfrac{1}{2}[C(1) - C'(-1)]. \tag{5.2}$$

This value for P and the relation between the curvatures and W' ensure that b vanishes at the poles. For example, let us consider

$$\eta(\cos\theta) = P_2(\cos\theta) - 1 - aP_3(\cos\theta), \tag{5.3}$$

where P_2, P_3 are Legendre polynomials. Note that P_2 has a symmetric squashing effect while P_3 squashes the bottom more than the top (figure 6): if $a = 0$ then, to order ϵ, the surface Γ is an ellipsoid. The calculation of b is simplified by the fact that the Legendre polynomials are eigenfunctions of C':

$$\eta = P_n(\cos\theta) \Rightarrow C' = \tfrac{1}{2}(n-1)(n+2)P_n(\cos\theta).$$

For Γ to be admissible, a must be related to W' by (5.2), which gives $W' = 5a$.

From (5.1) we get

$$b^2 = 3(1 - \alpha\cos\theta)\sin^2\theta \quad \text{with} \quad \alpha = \tfrac{25}{6}a = \tfrac{5}{6}W'. \tag{5.4}$$

Note further that since $b(\theta)$ must be non-negative, we also require $\alpha < 1$.

Let us calculate $\delta^2 E$ for rigid displacements. In this case the only non-zero contribution is $\delta^2 E_m$. Now to leading order in ϵ, $\delta^2 E_m$ reduces to an integral over the sphere:

$$\delta^2 E_m = \frac{2\gamma}{R}\int_\Gamma \left(-\frac{1}{R}\tau_n^2 b^2 - \tau_n \varphi \frac{\partial \varphi}{\partial r}\right) R^2 \sin\theta \, d\theta \, d\phi, \tag{5.5}$$

where again we have used (3.5) to rewrite the compression term. To leading order the boundary condition (3.5*b*) can also be taken on the sphere, so that

$$\frac{\partial \varphi}{\partial r} = \Psi(\theta, \phi) \equiv \frac{1}{R \sin\theta} \frac{\mathrm{d}(\sin\theta \tau_n b)}{\mathrm{d}\theta} \tag{5.6}$$

(we have taken b instead of B, so that ϕ is conveniently rescaled). For general rigid displacements, we may set

$$\tau_n = Y_1^0(\theta, \phi)\, h_v - (Y_1^1(\theta, \phi) - Y_1^{-1}(\theta, \phi))\, h_r, \tag{5.7}$$

where the $Y_n^m(\theta, \phi)$ are the complex spherical harmonics:

$$Y_1^0(\theta, \phi) = \frac{\cos\theta \sqrt{3}}{2\pi^{1/2}}; \quad Y_1^1(\theta, \phi) = \frac{\mathrm{e}^{\mathrm{i}\theta} \cos\phi \sqrt{6}}{4\pi^{1/2}},$$

and h_v, h_r are constants which denote respectively the vertical and horizontal displacements. Using expression (5.4) for b^2 and our choice for τ_n, the first term in (5.5) (the curvature term) integrates to

$$-\tfrac{24}{5} h_r^2 - \tfrac{6}{5} h_v^2.$$

To evaluate the compression term we must solve (3.5). To leading order we can express the solution φ in terms of spherical harmonics. By injecting this series solution into the compression term and integrating term by term we obtain the series

$$\sum_{k=0}^{\infty} \frac{1}{k+1} \left(|\psi_{k,0}|^2 + 2 \sum_{j=1}^{k} |\psi_{k,j}|^2 \right)$$

(all cross-terms vanish due to the orthogonality of the spherical harmonics) where the $\psi_{k,j}$ are the harmonic coefficients of the right-hand side of (5.6):

$$\psi_{k,j} = \sum_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \overline{Y_k^j(\theta, \phi)}\, \Psi(\theta, \phi) \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi.$$

It is easily verified that Ψ has the form

$$\Psi(\theta, \phi) = F(\theta)\, h_v + G(\theta) \cos\phi h_r,$$

so that the only non-zero coefficients will be those with $j = 0, 1$. These have the form

$$\psi_{k,0} = F_k(\alpha)\, h_v, \quad \psi_{k,1} = G_k(\alpha)\, h_r.$$

Hence summing the surface and compression terms we obtain $\delta^2 E_m$ in the form

$$\delta^2 E_m = M_v(\alpha)\, h_v^2 + M_r(\alpha)\, h_r^2,$$

with $$M_v(\alpha) = -\frac{24}{5} h_r^2 + \sum_{k=0}^{\infty} \frac{|F_k(\alpha)|^2}{k+1}, \quad M_r(\alpha) = -\frac{6}{5} h_v^2 + 2 \sum_{k=1}^{\infty} \frac{|G_k(\alpha)|^2}{k+1}.$$

The sign of $\delta^2 E_m$ therefore depends on the functions M_v, M_r. For $\alpha \ll 1$, which gives almost ellipsoidal shapes, we can expand (5.4):

$$b = \sqrt{3}(1 - \tfrac{1}{2}\cos\theta\alpha + \tfrac{1}{8}\cos^2\theta\alpha^2 + \dots) \sin\theta.$$

To this order the $\psi_{k,j}$ vanish for k greater than 4, so that M_v, M_r can be calculated explicitly. These gruelling calculations were done automatically using MAPLE, a formal calculus computer package, which gave

$$\delta^2 E_m = \tfrac{6}{5}(h_v^2 - h_r^2) + (\tfrac{3}{350} h_v^2 + \tfrac{183}{700} h_r^2)\, \alpha^2 + \dots.$$

So it seems that these shapes are unstable under horizontal displacements and stable under vertical displacements.

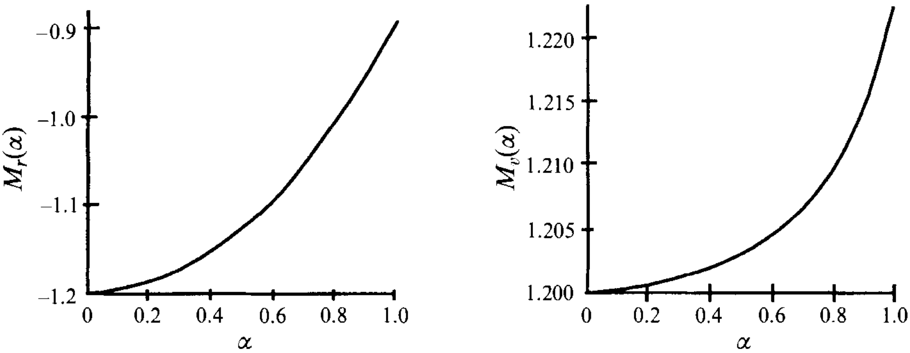

FIGURE 7. Graphs of $M_v(\alpha)$, $M_r(\alpha)$ for the feasible range of α. As $\delta^2 E_m = M_v(\alpha) h_v^2 + M_r(\alpha) h_r^2$, these show that this shape is unstable under rigid horizontal displacements.

Dropping the assumption of small α we can still integrate the $\psi_{k,j}$ explicitly, even if the resulting expressions are rather big (again, done using MAPLE). We find that for $k = 5$ the values of $F_k(\alpha)$, $G_k(\alpha)$ are already as small as 0.001. We can therefore safely truncate the series at $k = 5$ thus obtaining an approximation to $O(10^{-6})$. We will not rewrite the resulting truncated expressions for M_v, M_r as they would fill an entire page at least! We have however plotted these functions for the feasible range $[0, 1]$ of α (figure 7). A glance at these graphs tells us that the signs of M_v, M_r remain unchanged. *We therefore conclude that this class of shapes must be unstable.*

The following observations can be made:

(i) It is interesting to note that rigid displacements have the opposite effect in the above two- and three-dimensional examples: here h_v is stabilizing while it is destabilizing for the two-dimensional example (vice versa for h_r). This difference must be somehow related to the different dimensions of the two problems.

(ii) We have implicitly assumed in our calculation of E_m that the surface magnetic field is axisymmetric. In this case the equilibrium equation (1.2) yields a unique analytic solution (up to a sign), as the pressure constant P is fixed by the condition that $\boldsymbol{B}$ must vanish at least one point on the surface (at the poles in our example). However (1.2) only depends on the modulus of $\boldsymbol{B}$. Hence, unlike in the previous two-dimensional example, even if P is uniquely determined one could still ask if there exist other (non-axisymmetric) fields satisfying (1.2). These could then give a different and possibly positive $\delta^2 E_m$. However this cannot be done since, as shown by Felici & Brancher (1991*a*), there must be at most one solution to (1.2) for any shape with spherical topology (the same is not true for the torus)!

6. Stabilizing effect of an outer conducting shield

If we find that a given shape is unstable we might try to stabilize it by varying the position of the inductors. However, the applied current distribution $\boldsymbol{j}$ does not appear explicitly in $\delta^2 E$, and so it will only affect $\delta^2 E$ in so far as it changes the magnetic field on the liquid surface. In fact the same liquid shape can be formed, at least in principle, by a current distribution which generates a different surface field, but which still satisfies the equilibrium equation (1.2). We would therefore hope in this way to determine a current distribution in the induction coils (done by solving the inverse problem) which renders $\delta^2 E$ positive.

This possibility is therefore tied to (1.2) admitting multiple solutions for the surface

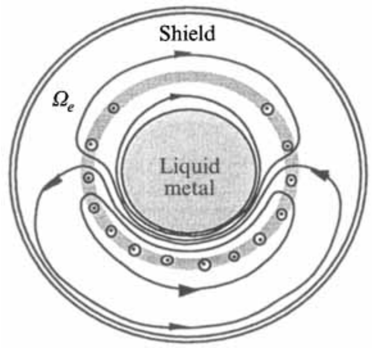

FIGURE 8. The presence of a metal shield has the effect of containing the generated magnetic field within itself.

field. This point was developed in Felici (1991) (see also Felici & Brancher 1991*a*), showing that the number of solutions to (1.2) depends on the topology of the liquid shape. In particular, as mentioned in the previous section, spherical shapes admit at most one solution, thus making it impossible to stabilize an unstable shape in this manner. On the other hand, tori allow multiple solutions, although the first example shows that it may not be possible to stabilize these shapes just by varying the levitating field.

Professor H. K. Moffat (personal communication) has suggested that for cylindrical configurations a stabilizing effect might be achieved on cross-sectional displacements by enclosing both the liquid and the current-carrying coils in a metal shield (figure 8). Such a shield would prevent the magnetic field from escaping, so that the subsequent flux compression induced by any fluid movement would presumably discourage the liquid surface from approaching this shield.

We now investigate the stabilizing effect of such a shield for the different topologies considered up to now (two-dimensional, spherical and toroidal). We show that the shield does indeed have a stabilizing effect not only on two-dimensional configurations (which seems plausible), but also for more general shapes.

The addition of this shield has the effect of increasing the compression term in $\delta^2 E_m$, while all the other terms in $\delta^2 E$ remain unchanged as they do not depend on the exterior domain. In the presence of such a shield (3.11) remains the same (the calculations leading to this being virtually the same), except that now Ω_e is the region between the liquid surface Γ and the shield Q. Also, in the case of a torus, the shield introduces a new toroidal discontinuity in the field potential with magnitude $I_{Q_{twist}}$ which is, as the name suggests, the total current spiralling round the shield. This leads to more terms appearing in the evaluation of (3.11). Again, these fortunately cancel out using the variation (as Γ moves) of the second consistency condition (again, appearing due to the presence of the shield):

$$\int_{\Lambda 1} \boldsymbol{B}\cdot\boldsymbol{n}_{\Lambda 1}\,\mathrm{d}S = 0 \tag{6.1}$$

where $\Lambda 1$ is any surface spanning the cross-section between Γ and Q (figure 9). The derivation of (6.1) is the same as for the first consistency condition (2.4).

Let us consider how the compression term varies as we change the shape of the shield. Let γ be an infinitesimal deformation of Q; then

$$\delta_\gamma \int_{\Omega_e} v^2\,\mathrm{d}V = 2\int_{\Omega_e} \boldsymbol{v}\cdot\delta_\gamma \boldsymbol{v}\,\mathrm{d}V - \int_Q v^2\gamma_n\,\mathrm{d}S, \tag{6.2}$$

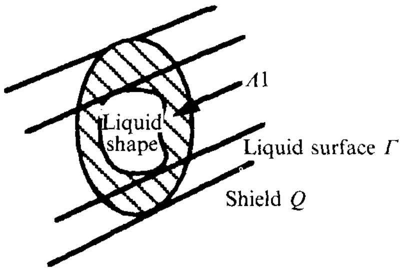

FIGURE 9. The surface of discontinuity $A1$.

where for simplicity we have set $\boldsymbol{v} = \delta \boldsymbol{B}$, $\gamma_n = \gamma \cdot \boldsymbol{n}$, and to avoid confusion with the variations of the liquid surface, the suffix γ denotes the variations induced by γ. Note that we take $\boldsymbol{n}$ as being the inward normal on Q, hence the minus sign. Equations (3.4) for $\delta \boldsymbol{B}$ are still the same, except that the extra condition $\boldsymbol{n} \cdot \boldsymbol{B} = 0$ is added as $\boldsymbol{B}$ does not penetrate Q. Since Q remains fixed as the liquid moves, we have $0 = \mathrm{D}(\boldsymbol{n} \cdot \boldsymbol{B}) = \boldsymbol{n} \cdot \delta \boldsymbol{B}$ on Q. Hence (3.4) becomes

$$\nabla \cdot \boldsymbol{v} = \nabla \wedge \boldsymbol{v} = 0 \quad \text{in} \quad \Omega_e. \tag{6.3a}$$

$$\boldsymbol{n} \cdot \boldsymbol{v} = \nabla_s \cdot (\tau_n \boldsymbol{B}) \quad \text{on} \quad \Gamma, \tag{6.3b}$$

$$\boldsymbol{n} \cdot \boldsymbol{v} = 0 \quad \text{on} \quad Q. \tag{6.3c}$$

The equations for $\delta_\gamma \boldsymbol{v}$ are derived from (6.3) in a similar manner:

$$\nabla \cdot \delta_\gamma \boldsymbol{v} = \nabla \wedge \delta_\gamma \boldsymbol{v} = 0 \quad \text{in} \quad \Omega_e, \tag{6.4a}$$

$$\boldsymbol{n} \cdot \delta_\gamma \boldsymbol{v} = 0 \quad \text{on} \quad \Gamma, \tag{6.4b}$$

$$\boldsymbol{n} \cdot \delta_\gamma \boldsymbol{v} = \nabla_s \cdot (\gamma_n \boldsymbol{v}) \quad \text{on} \quad Q. \tag{6.4c}$$

The derivation of (6.4*c*) is the same as for (3.4*b*) since $\boldsymbol{v}$ is tangent to Q. On the other hand (6.4*b*) is derived by the requirement to keep the field on the surface independent of the position of the shield. This is necessary as we intend to analyse the stability for a given predefined shape, which entails finding the corresponding levitating currents (done by solving the inverse problem) which will of course vary with the position of the shield.

If Γ is spherical or two-dimensional, we have that $\boldsymbol{v} = \nabla \phi$, so by integration by parts and using (6.4*a*):

$$\int_{\Omega_e} \boldsymbol{v} \cdot \delta_\gamma \boldsymbol{v} \, \mathrm{d}V = \int_{\Omega_e} \nabla \cdot (\phi \delta_\gamma \boldsymbol{v}) \, \mathrm{d}S = -\int_{Q+\Gamma} \phi \boldsymbol{n} \cdot \delta_\gamma \boldsymbol{v} \, \mathrm{d}S$$

$$= -\int_Q \phi \nabla_s \cdot (\gamma_n \boldsymbol{v}) \, \mathrm{d}S$$

using (6.4*c*);

$$= \int_Q \nabla_s \phi \cdot \boldsymbol{v} \gamma_n \, \mathrm{d}S \tag{6.5}$$

integrating by parts on Q. Hence substituting the above result in the first term of (6.2) and summing the two we have the result:

$$\delta_\gamma \int_{\Omega_e} v^2 \, \mathrm{d}V = + \int_Q v^2 \gamma_n \, \mathrm{d}S. \tag{6.6}$$

This equation also applies to toroidal shapes: as in the derivation of (3.11), extra terms appear when integrating by parts owing to the discontinuities in φ. But these cancel out

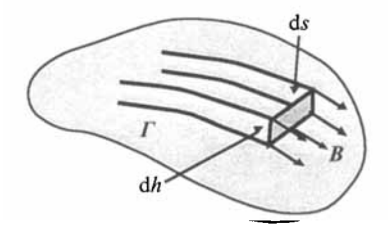

FIGURE 10. Diagram illustrating the 'pill box' argument on the surface field.

using the variation (as Q moves) of the consistency conditions (2.4), (6.1). Hence providing (6.2) has a non-trivial solution in Ω_e ($\boldsymbol{v} \neq \boldsymbol{0}$), the compression term increases if γ_n is positive, which corresponds to contracting the shield Q uniformly (note that, as ϕ is a non-constant harmonic function, $\boldsymbol{v} = \boldsymbol{\nabla}\phi$ cannot vanish everywhere on Q, thus showing that the variation of the compression term is strictly positive).

So the shield increases the stability for a given perturbation τ_n, but for the shield to be able to stabilize an unstable perturbation, we need to show that we can make the compression term sufficiently large so as to make $\delta^2 E$ positive. Indeed this seems to be the case, but before justifying this somewhat stronger affirmation, we derive some important consequences for the relationship between stability and the topology of the liquid shape.

6.1. *The stabilizing effect of the shield for different topologies*

We have shown that the shield has a stabilizing effect on any perturbation τ_n for which the solution to (6.3) is non-trivial. This is always the case unless τ_n satisfies

$$\boldsymbol{\nabla}_s \cdot (\tau_n \boldsymbol{B}) = 0 \tag{6.7}$$

on the entire liquid surface, in which case the compression term is zero independently of the position of the shield. Therefore the shapes that can be stabilized by the addition of the shield will be those for which (6.7) admits no solution for τ_n. Hence we can use (6.7) to analyse the stability for a variety of shape topologies and field configurations. The analysis is simplified by noting the following simple geometric interpretation of (6.7): consider a small (three-dimensional) magnetic flux tube lying on the liquid surface (figure 10), then, since $\boldsymbol{B}$ is tangent to the surface, from $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ we must have $B\,\mathrm{d}h\,\mathrm{d}s = \text{const.}$ where $\mathrm{d}h$ is the height and $\mathrm{d}s$ is the width of the flux tube. But by surface flux conservation, (6.7) implies $\tau_n B\,\mathrm{d}s = \text{const.}$ so that dividing the two we get

$$\tau_n = k\,\mathrm{d}h, \tag{6.8}$$

where k is constant along the surface field line. It follows that surface deformations which leave the field unchanged are those that 'follow the field lines neighbouring the surface'.

We can investigate the existence of such unwelcome disturbances by simple topological arguments:

(i) *Two-dimensional shapes* For two-dimensional perturbations, condition (6.7) implies by simple integration that $\tau_n B$ is constant. But as τ_n must be volume preserving it must change sign at some point, so $\tau_n B$ must vanish on the entire surface. As B is analytic, it can only be zero at a finite number of points which implies that no non-trivial (analytic) τ_n can satisfy the above condition. It follows that *all two-dimensional shapes can be stabilized for any two-dimensional smooth (analytic) disturbance.*

(ii) *Spherical topologies* In this case the surface field can be written as the surface gradient of a single-valued potential function: $\boldsymbol{B} = \boldsymbol{\nabla}_s \Phi$ (since $\boldsymbol{B}\cdot\boldsymbol{n} = 0$ on Γ). The

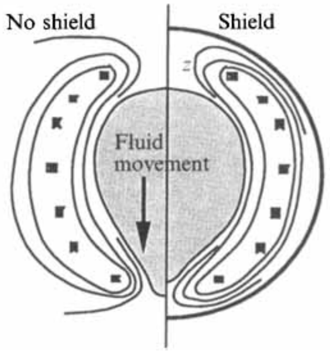

FIGURE 11. The presence of the metal shield has the effect of making the magnetic hole more rigid.

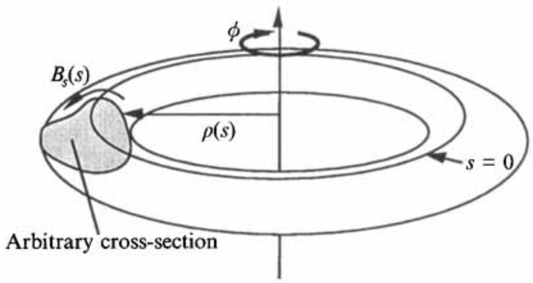

FIGURE 12. Diagram showing a toroidal shape with a poloidal field.

vector field $\tau_n \boldsymbol{B}$ must therefore have convergent fields lines where Φ reaches an isolated maximum or a minimum on Γ (of course, this must occur at at least two points). It follows that τ_n must vanish in an open neighbourhood of these points, otherwise by conservation of surface flux (a consequence of null surface divergence) $\tau_n \boldsymbol{B}$ would have infinite magnitude where the field lines converge. By flux conservation this trivial solution must therefore propagate to the entire surface. Hence *the shield has a stabilizing effect on all spherical shapes.*

Note in particular that the increased flux compression due to the presence of the shield has physically the effect of making the 'magnetic hole more rigid', thus discouraging the liquid from falling through it (figure 11).

(iii) *Toroidal topologies* Toroidal shapes allow some degree of freedom in the choice of surface fields. Consequently they provide a richer variety of stability behaviour. In fact we will see that perturbations which cannot be stabilized by the addition of the shield (i.e. satisfying (6.7)) may be allowed depending on the structure of the surface magnetic field lines. This is because this topology does admit surface $\boldsymbol{B}$-fields with no isolated singular points, as Φ is allowed to have multiple values.

For example consider any axisymmetric torus formed by an axisymmetric coil arrangement. The field $\boldsymbol{B}$ will therefore lie along the lines of symmetry, so that if $\hat{\boldsymbol{e}}_s$ is the surface unit vector and s is the distance parameter in the poloidal direction (figure 12) then $\boldsymbol{B} = B_s(s)\hat{\boldsymbol{e}}_s$, where $B_s(s)$ is given by the equilibrium equation (1.2). In this case, $\nabla_s \cdot (\tau_n \boldsymbol{B})$ becomes

$$\frac{1}{\rho(s)} \frac{\partial(\rho(s)\,\tau_n B_s)}{\partial s}.$$

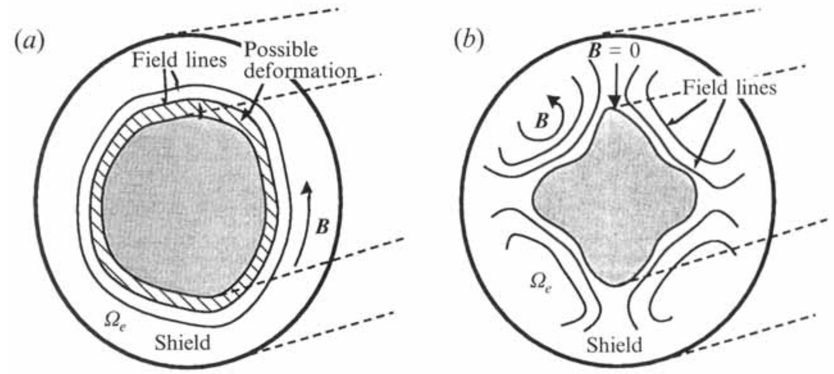

FIGURE 13. Assuming a poloidal field on the surface of the torus, it is clear from (6.8) that a disturbance leaving the field unaltered in Ω_e is always possible in case (*a*) but never in the case (*b*). The shield will therefore have no stabilizing effect on the disturbance in case (*a*).

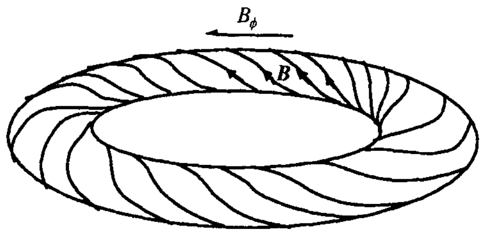

FIGURE 14. The addition of a twist in the poloidal field lines on the torus has a stabilizing effect, as the twist destroys any existing solutions to (6.7).

Hence τ_n satisfies $\nabla_s \cdot (\tau_n \boldsymbol{B}) = 0$ only if

$$\tau_n(s, \varphi) = \frac{f(\phi)}{\rho(s)\, B_s(s)},$$

$f(\phi)$ being any function which makes $\tau_n(s, \phi)$ volume preserving (i.e. $\int_0^{2\pi} f(\phi)\, \mathrm{d}\phi = 0$).

Several possibilities therefore arise.

(*a*) If $B_s(s)$ is everywhere positive, as in the case of a circular-cross section of the first example, then such τ_n clearly exist (figure 13*a*). Therefore the shield, although stabilizing cross-sectional perturbations, cannot eliminate these toroidally varying disturbances. Moreover for tori with convex cross-sections these perturbations give $\delta^2 E_m < 0$, since the curvature term is negative. If we add the destabilizing pinch effect due to surface tension we conclude that these surfaces are generally unstable under longitudinal perturbations.

It should be noted that in practice we may overcome this difficulty by applying a strong steady toroidal (longitudinal) field, which, although not providing any levitation force, is known to have the effect of damping any fluid motion normal to its direction. This would then discourage the onset of longitudinal instabilities in the toroidal liquid shape.

(*b*) If the cross-sections considered have $B_s(s)$ vanishing on some line of symmetry, then τ_n will be well defined only if $f(\phi)$ vanishes. Consequently this type of $\boldsymbol{B}$-field does not admit any such perturbation. The shield will therefore have a stabilizing effect on all possible disturbances! Shercliff's well-known cruciform cross-sections are an example of such 'stabilizable' shapes (figure 13*b*).

(*c*) More exotic behaviour appears if we add a longitudinal twist to the levitating field by applying a constant longitudinal field component B_ϕ (figure 14). It turns out that this has a stabilizing effect on the shape in case (*a*) (i.e. where the surface field has no singular points) by destroying any existing solutions to (6.7). The component B_ϕ can be generated by adding an induction coil wound round the torus with a current at the same frequency as that in the initial coil. The 'twist' on the magnetic field will then be determined by the current intensity in these coils. We may adjust the currents so that the field lines 'never join up' after one or several revolutions round the torus (i.e. the twist number is irrational; it would of course be very difficult to obtain a rational twist number!). Hence all the field lines are the continuation of one single field line. So by flux conservation τ_n satisfying $\nabla_s \cdot (\tau_n \boldsymbol{B}) = 0$ must have constant sign on Γ. But this cannot exist as it would not be volume preserving.

So the field twist offers a way of stabilizing surfaces such as the torus in the first example. It is worth noting that the stability increases with the intensity of B_ϕ (even in the absence of the shield), since in this case

$$\delta^2 E_m = -\frac{1}{\mu_0}\int_\Gamma \tau_n^2 (C_s B_s^2 + C_\phi B_\phi^2)\,\mathrm{d}S + \frac{1}{\mu_0}\int_{\Omega_e} |\delta \boldsymbol{B}|^2\,\mathrm{d}V,$$

where in the curvature we have written $\kappa_B B^2$ using the principal curvatures C_s, C_ϕ of the axisymmetric torus. In the limit as the torus radius $R_\tau \to \infty$, C_s becomes the cross sectional curvature, while $C_\phi \to 0$. In this limit the curvature term is independent of B_ϕ, while the compression increases like B_ϕ^2, due to the dependence of $\delta \boldsymbol{B}$ on $\boldsymbol{B}$ through (6.3*b*). This confirms the idea proposed by Sneyd & Moffatt (1982) that such toroidal forms may be stabilized by applying a longitudinal high frequency field.

6.2. *When does the compression term tend to infinity?*

As remarked earlier, for the shield to stabilize a given shape we need to show that the compression term increases sufficiently as the space between the shield and the liquid surface is reduced. Indeed it seems in general that the compression term increases without limit as the shield approaches the liquid surface! For example suppose that Γ is a sphere. If φ is the potential of $\delta\boldsymbol{B}$, then (6.3) is just

$$\nabla^2 \varphi = 0 \quad \text{in} \quad \Omega_e, \tag{6.9a}$$

$$\frac{\partial \varphi}{\partial n} = \Psi \quad \text{on} \quad \Gamma, \tag{6.9b}$$

$$\frac{\partial \varphi}{\partial n} = 0 \quad \text{on} \quad Q. \tag{6.9c}$$

Without loss of generality we may choose G with radius 1 and the shield Q as being a concentric sphere of radius R. In this case we can express the solution φ as a series expansion in spherical harmonics:

$$\varphi(r,\theta,\phi) = \sum_{n=1}^{\infty}\sum_{n=-m}^{m}\left(A_{n,m} r^n + \frac{B_{n,m}}{r^{n+1}}\right) Y_n^m(\theta,\phi), \tag{6.10}$$

with $$A_{n,m} = -\frac{\psi_{n,m}}{n(R^{2n+1}-1)}, \quad B_{n,m} = -\frac{R^{2n+1}\psi_{n,m}}{(n+1)(R^{2n+1}-1)}.$$

Substituting (6.10) into the compression term (see (3.11)) and integrating term by term we obtain

$$\int_{\Omega_e} |\nabla\varphi|^2\,\mathrm{d}V = \sum_{n=1}^{\infty} \frac{n+1+nR^{2n+1}}{n(n+1)(R^{2n+1}-1)}\left(|\psi_{n,0}|^2 + 2\sum_{m=1}^{n}|\psi_{n,m}|^2\right)$$

(all cross-terms vanish due to the orthogonality of the spherical harmonics), where the $\psi_{n,m}$ are the harmonic coefficients of Ψ (note that Ψ must have vanishing surface integral, so $\Psi_{0,0} = 0$). It is clear that if at least one of the harmonic components of Ψ is non-zero then

$$\int_{\Omega_e} |\nabla\varphi|^2 \, dV \sim \frac{1}{d} \to \infty$$

as the distance $d = R - 1$ between Γ and Q tends to zero.

An analogous result to the above is obtained for toroidal shapes. If we consider Γ as being a large torus with circular cross-section, then Γ can be taken as a circular cylinder on which $\delta \boldsymbol{B}$ is periodic in the longitudinal direction (the period p_T corresponding to the perimeter of the torus). The corresponding potential function φ generally has an increase $\delta I_{Q_{twist}}$ (the variation of the total current spiralling round the shield) in the longitudinal direction every period p_T, and a constant increase δI_Γ every cross-sectional revolution. Both $\delta I_{Q_{twist}}$ and δI_Γ are constant, which enables us to write the general solution to (6.3) as a discrete sum involving Bessel functions of second type. A similar expression to the above is obtained, having the same asymptotic behaviour for any non-zero Ψ.

7. Remarks and conclusions

The above results can be summarized in the following way.

(i) The stability properties of equilibrium configurations for liquid shapes are tied to condition (6.7): the shape may be made stable by the presence of a surrounding metal shield only if the topologies of both the surface magnetic field and of the shape are such that no regular perturbation is admitted satisfying (6.7). Physically this means that no perturbation can leave the exterior magnetic field unchanged.

(ii) From a simple geometrical interpretation (6.8) and (6.7) the following remarks can be made for various topologies: for two-dimensional configurations in general, all shapes can be stabilized by the addition of a shield for any two-dimensional smooth (analytic) disturbance. The same turns out to be true for all three-dimensional spherical topologies.

(iii) For the toroidal case, several situations arise depending on the shape of the surface magnetic field, as some surface field topologies seem to allow perturbations satisfying (6.7) (figure 13). However if a 'twist' B_ϕ (figure 14) is added to the magnetic field lines, not only are the destabilizing perturbations eliminated, thus making all toroidal shapes stabilizable by the addition of the shield, but the overall stability is increased as well, even for those configurations that were already stable in the first place.

Besides its stabilizing effect, the shield also has the advantage of preventing the escape of magnetic energy, thus increasing the efficiency of the levitation device. Another important advantage is that the shield increases the stability without having to increase the magnetic field intensity on the liquid surface, and with it the internal fluid motion (which would reduce the validity of our analysis). For example, as already mentioned, toroidal shapes can be stabilized by applying a strong longitudinal field B_ϕ. This has the disadvantage of increasing the surface field. The same stability may be obtained by placing the shield sufficiently close to the liquid surface. We may then choose a levitating current distribution which only gives a weak B_ϕ on the liquid surface.

In practice we would like to place the shield as close as possible to the liquid surface.

However, there is the obvious restriction of the inductors, which have to remain at a safe distance from the molten metal. Moreover the levitating current distribution $\boldsymbol{j}_S$ is given by

$$\boldsymbol{j}_S = \boldsymbol{n}_S \wedge \Delta \boldsymbol{B},$$

where $\boldsymbol{n}_S$ is the normal to the inductor surface S and $\Delta \boldsymbol{B}$ is the jump in the magnetic field across S. The order of magnitude of $\boldsymbol{j}_S$ must therefore increase without limit as Q approaches S in order to compensate for the field deformation between the two surfaces S and Q. Hence the total power needed to generate the levitating currents, proportional to $\int_S j_S^2 \,\mathrm{d}S$ therefore tends to infinity. Hence we cannot place the shield just behind the surface inductor in an attempt to maximize the stabilizing action of the shield: in practice there has to be a trade-off between optimum stability and the power we are prepared to inject into the system.

I would like to thank J. P. Brancher, as well as the referees, for the helpful remarks made during the writing of this paper.

Appendix

We remind the reader of the following properties we need in these evaluations: Let $\boldsymbol{T}'_\epsilon(\boldsymbol{x}_0)$ be the matrix with components $\partial T^i_\epsilon/\partial x^j_0$. Then we have

$$\mathrm{d}\boldsymbol{T}'_\epsilon/\mathrm{d}\epsilon = \tau'_\epsilon, \tag{A 1}$$

where $\boldsymbol{\tau}'_\epsilon$ is the Jacobian matrix with components $\partial \tau^i_\epsilon/\partial x^j_0$. Now let $\boldsymbol{Y}_\epsilon = \boldsymbol{T}'^{-1}_\epsilon$. Then $\boldsymbol{Y}_\epsilon \boldsymbol{T}'_\epsilon = \boldsymbol{I}$, the identity matrix. Hence differentiating and using (A 1):

$$\boldsymbol{Y}_\epsilon \boldsymbol{\tau}'_\epsilon + \frac{\mathrm{d}\boldsymbol{Y}_\epsilon}{\mathrm{d}\epsilon} \boldsymbol{T}'_\epsilon = 0,$$

and so

$$\frac{\mathrm{d}\boldsymbol{T}'^{-1}_\epsilon}{\mathrm{d}\epsilon} = -\boldsymbol{T}'^{-1}_\epsilon \boldsymbol{\tau}'_\epsilon \boldsymbol{T}'_\epsilon \tag{A 2}$$

A.1. *Evaluation of $\delta \boldsymbol{n}$*

We have $\boldsymbol{n} = \nabla \Psi / |\nabla \Psi|$ where $\Psi(x) = 0$ is a local representation of Γ. But

$$\nabla \Psi = {}^t T'^{-1}_\epsilon \frac{\partial \Psi_0}{\partial \boldsymbol{x}_0}$$

where $\Psi_0(\boldsymbol{x}_0) = \Psi(T^{-1}(\boldsymbol{x}))$ (the superscript t denotes the transpose), which implies that $\delta(\nabla \Psi) = -{}^t\tau' \,\partial \Psi/\partial \boldsymbol{x}$ from (A 2), and

$$\delta|\nabla \Psi| = \delta(\nabla \Psi \cdot \nabla \Psi)^{1/2} = \frac{\nabla \Psi \cdot \delta(\nabla \Psi)}{|\nabla \Psi|} = -\frac{1}{|\nabla \Psi|}\left\langle \frac{\partial \Psi}{\partial x}, {}^t\tau' \frac{\partial \Psi}{\partial x} \right\rangle,$$

so

$$\delta \boldsymbol{n} = \frac{\delta(\nabla \Psi)}{|\nabla \Psi|} - \frac{\delta|\nabla \Psi|}{|\nabla \Psi|^2} \nabla \Psi = -{}^t\tau' \boldsymbol{n} + \langle n, {}^t\tau' n \rangle \boldsymbol{n}.$$

Hence, in component language,

$$\delta n^i = -\partial_i \tau^k n^k + (n^j \partial_j \tau^k n^k) n^i$$

which with $\tau_n = \tau^k n^k$

$$= -\partial_i \tau_n + \tau^k \partial_i n^k + (n^j \partial_j \tau_n - \tau^k n^j \partial_j n^k) n^i$$

$$= -(\partial_i - n^i n^j \partial_j) \tau_n + \tau^k (\partial_i - n^i n^j \partial_j) n^k$$

which by definition of $\boldsymbol{d}$

$$= -\mathrm{d}_i \tau_n + \tau^k \mathrm{d}_i n^k$$

and by symmetry of $\boldsymbol{dn}$

$$= -\mathrm{d}_i \tau_n + \tau^k \mathrm{d}_k n^i$$

Hence

$$\delta \boldsymbol{n} = -\boldsymbol{\nabla}_s \tau_n + (\tau_t \cdot \boldsymbol{\nabla}_s)\, \boldsymbol{n}$$

A.2. *Evaluation of* δC

For simplicity we shall only derive δC for normal displacements. These deviations are not new: for a more general study see Bang-Yen Chen (1984, Chap. 5.4, p. 213). First, note that we can rewrite $\boldsymbol{dn}$, the second fundamental form, in terms of then principal directions $\boldsymbol{t}_1\, \boldsymbol{t}_2$ with respective principal curvatures κ_1, κ_2 at a point on Γ. Then

$$\mathrm{d}_i n^j = \kappa_1 t_1^i t_1^j + \kappa_2 t_2^i t_2^j. \tag{A 3}$$

It follows that the mean curvature $C \equiv \kappa_1 + \kappa_2$ is given by

$$C = \mathrm{d}_k n^k$$

since $\boldsymbol{t}_1$, $\boldsymbol{t}_2$ are unit vectors. The evaluation of δC is simplified by noting that $\mathrm{d}_k n^k = \partial_k n^k$. Hence

$$\delta C = \delta(\partial_k n^k) = \delta(\partial_k)\, n^k + \partial_k(\delta n^k)$$

if we only consider normal displacements, i.e. $\boldsymbol{\tau} = \tau_n \boldsymbol{n}$, then, from above, $\delta \boldsymbol{n} = -\boldsymbol{d}\tau_n$ and so the normal variation is given by, since as above $\delta\, \partial/\partial \boldsymbol{x} = -{}^t\tau'\, \partial/\partial \boldsymbol{x}$,

$$\begin{aligned}\delta_n C &= -\partial_k \tau^i \partial_j n^k - \partial_k \mathrm{d}_i \tau_n \\ &= -\partial_k(\tau_n n^j)\, \partial_j n^k - \mathrm{d}_i \mathrm{d}_i \tau_n - n^i \frac{\partial}{\partial n} \mathrm{d}_i \tau_n \\ &= -\partial_k \tau_n \frac{\partial n^k}{\partial n} - \tau_n \partial_k n^j \partial_j n^k - d_i d_i \tau_n - n^i \frac{\partial}{\partial n} d_i \tau_n.\end{aligned}$$

The first and last terms cancel because they sum to $(\partial/\partial n)\,(n^i \mathrm{d}_i \tau_n)$. Moreover it is easily verified that $\partial_k n^j \partial_j n^k = \mathrm{d}_k n^j \mathrm{d}_j n^k$ so that we are left with

$$\delta_n C = -\tau_n \mathrm{d}_k n^j \mathrm{d}_j n^k - \mathrm{d}_i \mathrm{d}_i \tau_n.$$

The first term is rewritten as (A 3):

$$\begin{aligned}\mathrm{d}_k n^j \mathrm{d}_j n^k &= \kappa_1^2 (\boldsymbol{t}_1 \cdot \boldsymbol{t}_1)\, (\boldsymbol{t}_1 \cdot \boldsymbol{t}_1) + \kappa_2^2 (\boldsymbol{t}_2 \cdot \boldsymbol{t}_2)\, (\boldsymbol{t}_2 \cdot \boldsymbol{t}_2) + 2\kappa_1 \kappa_2 (\boldsymbol{t}_1 \cdot \boldsymbol{t}_2)\, (\boldsymbol{t}_1 \cdot \boldsymbol{t}_2) \\ &= \kappa_1^2 + \kappa_2^2\end{aligned}$$

since $\boldsymbol{t}_1, \boldsymbol{t}_2$ are mutually orthonormal. Hence, noting that $\mathrm{d}_i, \mathrm{d}_i$ is just the surface Laplacian ∇_s^2, we finally have

$$\delta_n C = -(\kappa_1^2 + \kappa_2^2)\, \tau_n - \nabla_s^2 \tau_n.$$

A.3. *Evaluation of (2.6)*

Take skin depth into account. Select c' (figure 3) in the skin region, then (as $\boldsymbol{j} = \sigma \boldsymbol{E}$ if we neglect fluid movement)

$$\int_{c'} \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{l} = \sigma \int_{c'} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} \tag{A 4}$$

$$= \mathrm{i}\omega\sigma \int_{\Lambda} \boldsymbol{B} \cdot \boldsymbol{n}_{\Lambda}\, \mathrm{d}S = \frac{\mathrm{i}}{\mu_0 \delta^2} \int_{\Lambda} \boldsymbol{B} \cdot \boldsymbol{n}_{\Lambda}\, \mathrm{d}S.$$

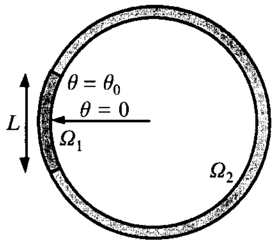

FIGURE 15. Division of Ω into Ω_1 and Ω_2.

If c is an adjacent curve running on the liquid surface Γ then

$$\int_{c'} \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{l} \sim \frac{1}{\delta} \int_{c} \boldsymbol{j}_\Gamma \cdot \mathrm{d}\boldsymbol{l},$$

$\boldsymbol{j}_\Gamma$ being the surface current density, and

$$\frac{1}{\mu_0 \delta^2} \int_{A'} \boldsymbol{B} \cdot \boldsymbol{n}_A \, \mathrm{d}S \sim \frac{1}{\mu_0 \delta^2} \int_{A} \boldsymbol{B} \cdot \boldsymbol{n}_A \, \mathrm{d}S + \frac{1}{\mu_0 \delta^2} \int_{c} \boldsymbol{B} \cdot \boldsymbol{n}_A \, \mathrm{d}S.$$

From (A 4) we therefore must have

$$\frac{1}{\delta} \int_{c} \boldsymbol{j}_\Gamma \cdot \mathrm{d}\boldsymbol{l} \sim \frac{1}{\mu_0 \delta^2} \int_{A} \boldsymbol{B} \cdot \boldsymbol{n}_A \, \mathrm{d}S + \frac{1}{\mu_0 \delta} \int_{c} \boldsymbol{B} \cdot \boldsymbol{n}_A \, \mathrm{d}S,$$

which can only be satisfied in the limit as $\delta \to 0$ if the $1/\delta^2$ coefficient tends to zero like $o(\delta)$.

A.4. *Calculation of (2.6) for a large torus*

$$\int_{A} \boldsymbol{B} \cdot \boldsymbol{n}_A \, \mathrm{d}S = \int_{c} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l},$$

where c is any curve running on the toroidal surface Γ and $\boldsymbol{A}$ is the vector potential:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{4\pi} \int_{\Omega} \frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \mathrm{d}^3\boldsymbol{r}'$$

where $\boldsymbol{j}$ is the current density in the current region Ω (i.e. the liquid torus plus the current carrying inductor coils). To estimate $\boldsymbol{A}$ at a given point $\boldsymbol{r}_0$, we divide Ω into two parts: a local part Ω_1 of fixed length L containing $\boldsymbol{r}_0$, and Ω_2 the rest (figure 15). The contribution of Ω_1 to $\boldsymbol{A}$ is finite if L remains finite. If the radius R_τ of the torus Γ is large compared to its cross-section, for points on the torus far from $\boldsymbol{r}_0$ the surface Γ and the inductor coils may be considered as a single wire $\boldsymbol{l}$ carrying the total current $I_\Gamma + I_S$. Hence the contribution of Ω_2 is approximately given by the line integral

$$\boldsymbol{A}_2 \approx \frac{I_\Gamma + I_S}{4\pi} \int_{\Omega_2} \frac{\mathrm{d}\boldsymbol{l}}{|\boldsymbol{r}_0 - \boldsymbol{r}'|}.$$

By symmetry, only the component of $\boldsymbol{A}_2$ along the 'wire' is non-zero, which can be written as

$$\frac{I_\Gamma + I_S}{4\pi} \int_{\theta_0}^{2\pi - \theta_0} \frac{\mathrm{d}\theta}{2\,|\sin(\theta/2)|},$$

where $\theta = 0$ is the position of $\boldsymbol{r}_0$ and θ_0 is the boundary between Ω_1 and Ω_2, so that $\theta_0 = L/(2R_\tau)$. This evaluates to

$$\frac{I_\Gamma + I_S}{2\pi} \ln \tan \frac{\theta_0}{4} = -\frac{I_\Gamma + I_S}{2\pi} \ln \tan \frac{L}{8R_\tau}.$$

As we let R_τ tend to infinity while keeping L fixed, θ_0 tends to zero, so that the above becomes the main contribution to $\boldsymbol{A}$. Hence we have

$$\int_c \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} \sim 2\pi R_\tau \frac{I_\Gamma + I_S}{2\pi} \ln \frac{8R_\tau}{L} \sim (I_\Gamma + I_S) R_\tau \ln R_\tau.$$

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